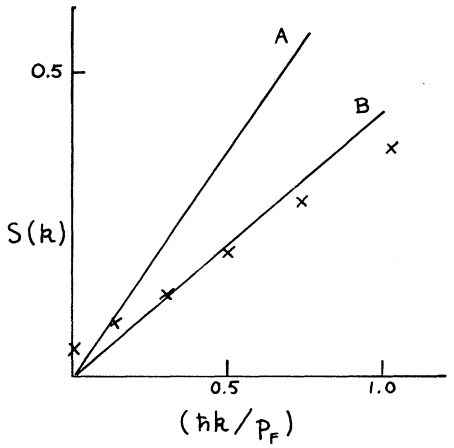

FIG. 1. Line B is the prediction of the Landau theory for the structure factor $S(k)$, i.e., $S(k)=\alpha(3\hbar k/4P_F)$. α is determined to be 0.575 for He^3. Line A indicates the structure factor for an ideal Fermi gas ($\alpha_{ideal}=1$). The experimental points are from the data of Ref. 2.

determined from a knowledge of the compressibility and the speed of zero sound, and F_l is assumed to vanish for $l \geq 2$. We have used neither the theory nor the data for the specific heat (the theory being quite controversial) in our considerations. It seems to us that our result constitutes strong evidence for the validity of the Landau theory of density oscillations in Fermi liquids.

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THICKNESS OF THE FLOWING SUPERFLUID FILM*

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It has recently been shown experimentally that the thickness of a saturated helium film is independent of its state of superflow. We present a simple model in which the thickness of such a film is fixed by dynamical considerations at the gas-film interface, and the required balance of forces is then provided by relatively minor adjustments in the film surface tension.

In the two-fluid hydrodynamics of He II, steady-state flow is governed by the equation

$$\nabla(\mu+\tfrac{1}{2}u_s^2)=0, \tag{1}$$

where μ is the chemical potential and u_s the superfluid velocity.[1] This is closely analogous to Bernoulli's equation in ordinary fluid mechanics. It means that, if u_s changes with position, some other quantity, such as the fluid pressure, must vary to change μ and balance Eq. (1). This is required since changes in u_s are accelerations in a frame moving with the superfluid, and need balancing forces.

Kontorovich[2] has predicted that in helium films the film thickness z would change in order to balance changes in u_s by means of the van der Waals force binding the film to the wall. Keller[3] has shown experimentally that the predicted changes in z do not occur. Huggins[4] has suggested that the temperature T varies instead in order to balance u_s by means of the thermomechanical effect, but this has also been found to be incorrect.[5] In the usual formulation of film hydrodynamics μ depends only on z and T, and so Eq. (1) appears to be violated.

We shall argue below that the difficulty arises from the common assumption that the film behaves just like the bulk liquid except for the van der Waals field, i.e.,

$$\mu=\mu_L-\alpha/z^3, \tag{2}$$

where μ_L is the chemical potential of the liquid and α is constant. This assumption becomes untenable when the surface tension γ, usually neglected in the liquid, becomes a more important source of forces than the pressure, that is, for thickness less than $z_0\approx\gamma/P\approx10^{-4}$ cm. The films of interest are in the range $z\lesssim5\times10^{-6}$ cm,[6] and $P\approx3\times10^3$ dyn/cm^2 is the vapor pressure of helium at ≈2°K.

We have analyzed the hydrodynamics and thermodynamics of the superfluid film, taking the point of view that a film in superflow may be regarded as a thermodynamic equilibrium state.[7] This was done by maximizing the entropy of the system consisting of wall + film + vapor, subject to the conservation of mass, momentum, and energy. The resulting conditions for equilibrium are these: The velocities of the wall, normal fluid, and gas must all be the same, and are henceforth taken equal to zero; the temperature must be uniform everywhere; the pressure P in the gas and film are equal at the interface; and

$$\mu + \tfrac{1}{2}u_s^2 = \mu_g, \tag{3}$$

where μ_g is the chemical potential of the gas. For μ we find the identity

$$\rho d\mu = -d\gamma + z dP - S dT - \rho_n u_s du_s. \tag{4}$$

Aside from the term in du_s, which represents an extra degree of freedom owing to superfluidity, Eq. (4) is the usual form for the chemical potential of an adsorbed film.[8] ρ is the mass per unit area, S the entropy per unit area, and ρ_n/ρ the normal fraction.

Using Eqs. (3) and (4), with $\rho_s = \rho - \rho_n$, and the equilibrium conditions with $\mu_g = \text{const}$, changes in a saturated film on a vertical wall at constant T may be described as follows:

$$-\delta\gamma + z\delta P + \rho_s u_s \delta u_s + \rho g \delta\chi = 0, \tag{5}$$

$$(RT/P)\delta P - \delta(\alpha/z^3) + g\delta\chi = 0, \tag{6}$$

$$(RT/P_\infty)\delta P_\infty + g\delta\chi = 0. \tag{7}$$

Here g is the acceleration of gravity, χ the height above the liquid surface (we have included a term $g\chi$ in each chemical potential), R the gas constant, and P_∞ the pressure in the gas far from the wall. Eqs. (6) and (7) express the balance of chemical potential in the gas, which we have assumed to be ideal. We have also assumed that the van der Waals potential in the gas is of the form α/Z^3, where Z is the distance from the wall. Equations (5)-(7) may be expected to hold, of course, only when there is no dissipation in the flow.

It is instructive to obtain Kontorovich's prediction from this formalism. Using Eq. (2) with $d\mu_L = dP/\rho_b - (\rho_n/\rho)u_s du_s$ with $\rho_b = \rho/z$ and comparing with Eq. (4), we find (at constant T) $d\gamma = \rho d(\alpha/z^3) = -(3\rho\alpha/z^4)dz$. At fixed χ, we find, from Eq. (5), $-\delta(\alpha/z^3) + (z/\rho)\delta P + (\rho_b/\rho)u_s \delta u_s = 0$. But from Eq. (6), $\delta(\alpha/z^3) = (RT/P)\delta P \gg (z/\rho)\delta P$, so we have

$$3\rho\frac{\alpha}{z^4}\delta z + \frac{\rho_s}{\rho}u_s \delta u_s = 0, \tag{8}$$

which is Kontorovich's prediction for the change of z with u_s.

The effect of the approximation, Eq. (2), is to assume that the surface tension in the film differs from γ_L, the surface tension of the liquid, only through the effect of the van der Waals force. We can easily find the magnitude of this effect:

$$\gamma_L - \gamma = -\int_\infty^z (3\rho_b \alpha/Z^3)dZ = \tfrac{3}{2}\rho_b \alpha/z^2 \approx 3\times 10^{-3} \text{ cgs}$$

for a film with $z \approx 3\times 10^{-6}$ cm. Since $\gamma_L \approx 0.3$ cgs, we have $\Delta\gamma/\gamma \approx 10^{-2}$, that is, a change in the film thickness from infinity to 300 Å changes γ by 1%. Thus, Eq. (2) has been taken to mean that, van der Waals forces aside, the film and the liquid are so nearly identical that any difference changes the surface tension by much less than 1%.

Put in these terms, the assumption is clearly untenable. It must be remembered, for example, that γ_L is itself anomalous, decreasing with temperature while the bulk latent heat is rising at the temperatures of interest here.[9] If this behavior is due, as Atkins[10] suggests, to thermal excitation of capillary waves, we must certainly expect the temperature dependence, and hence γ itself, to be profoundly changed in thin films.

In this light, then, Keller's result for flowing films is not surprising. γ must adjust to balance Eq. (5), but it does so principally by changing properties other than z, upon which it is only weakly dependent. What is more surprising is that the thickness of the stationary film varies with χ approximately in the way predicted by Eq. (2), i.e., $g\chi = \alpha/z^3$. We wish now to present a model which has this feature.

We wish to assume that the rate of evaporation from a fairly thick film surface into a vacuum is the same as that from the liquid surface. This is a much less extreme assumption than Eq. (2),[3] and it seems especially sensible in the case of superfluid films. In this instance evaporation is thought to be due entirely to the normal fluid.[11] Thus we suppose that the rate of evaporation is a function only of the normal fluid fraction ρ_n/ρ. Since the normal fluid does not move, we do not expect this to change with u_s. Dynamic equilib-

rium at the interface requires that condensation from the gas take place at the same rate, and assuming a constant accommodation coefficient, this latter depends only on the local pressure (T being fixed). Thus P is a function only of ρ_n/ρ, so that for films thick enough to have ρ_n/ρ equal to the bulk liquid value (including the cases of interest), the local pressure at the interface is always equal to the saturated vapor pressure. Thus from Eq. (6) with $\delta P = 0$, we have immediately

$$g\chi - \alpha/z^3 = 0. \tag{9}$$

This equation therefore arises not from the correctness of Eq. (2), but merely from the rate of evaporation from the film surface, and from the properties of an ideal gas. The particular functional form α/z^3 is only approximate. However, we do predict (see below) that the rate of evaporation from a film will be found to be correlated with the dependence of z on χ (or on P_∞ in unsaturated films). Equation (9) must be thought of as a constraint on the properties of the film. We may now note that although from Eqs. (5) and (6) for $u_s = 0$, γ is still given by $\delta\gamma = \rho\delta(\alpha/z^3)$, this is an imposed condition, and may reflect considerable internal rearrangement of the film properties required to conform with Eq. (9).

There remains now only to suggest a possible mechanism by which γ may change in order to balance Eq. (5) for a flowing film while still obeying Eq. (9). The simplest suggestion is that γ should certainly be sensitive to the mean interatomic spacing, i.e., to $\rho_b = \rho/z$. If we suppose that this provides the required change, we deduce

$$\rho_s u_s \delta u_s = \delta\gamma = (\partial\gamma/\partial\rho_b)\delta\rho_b = (\partial\gamma/\partial\rho_b)z^{-1}\delta\rho.$$

It is actually the mass per unit area $\delta\rho$ which is measured in Keller's experiment. If we make the crude estimate $-\partial\gamma/\partial\rho_b \approx \gamma/\rho_b \approx 3.5$ cgs, we find that for a given change in u_s the change in ρ is roughly two orders of magnitude less than that expected from Kontorovich's prediction, Eq. (8). This estimate is within the experimental uncertainty reported by Keller.[3]

In summary, then, if we assume that the dynamic rate of evaporation from the surface of a film is the same as that from the surface of the liquid, Eq. (9) becomes the condition for dynamic equilibrium at the interface and fixes the thickness of the film regardless of its state of flow and any other internal properties. The surface tension which provides the principal force acting on the film is then very nearly equal to that of the liquid, not because the internal contributions to it are the same as those in the liquid—we expect them to be quite different—but because it is constrained to be so by the condition for dynamic equilibrium, and the internal contributions must adjust accordingly. Flow in the film leaves the thickness unchanged. The surface tension is obliged to change in order to balance Eq. (1), but it can easily do so by a relatively minor change in some internal variable such as the mean bulk density.

We may now turn the analysis around to make explicit predictions. We predict that a simple functional dependence of z on χ [or on P_∞ as in Eq. (7)] should obtain so long as the nature of evaporation from the film into a vacuum is the same as it is in the bulk liquid, and that the dependence of z on χ (or P_∞) should change when the evaporation does (say, for thin enough films). We also predict that the evaporation should be unaffected by superflow in the film. Measurements of evaporation from the bulk liquid have been made,[12] and similar measurements in the film should be feasible. We do not expect the instantaneous rate of evaporation from a film of fixed temperature and thickness to be an easily measurable quantity, but a corollary to our assumption is that the velocity distribution for atoms escaping from the surface of a stationary or moving film should be the same as that already observed from the liquid.[13]

In closing, we would like to comment on the broader significance of the idea we have presented here. Equation (2) is a form of the Frenkel-Halsey-Hill isotherm,[14] which is a cornerstone of the theory of adsorbed films. Based on this theory, Keller's experiment has been taken as a "clear contradiction"[5] of the hydrodynamic theory of He films. We have suggested instead that the hydrodynamic theory of He films is entirely consistent with experiment, that the special property of superflow has provided a new means of testing the Frenkel-Halsey-Hill idea, and that this has been found to be inadequate. Although it is not far off in predicting film thicknesses, the agreement is essentially fortuitous, and leads to serious errors in analyzing other film properties. The picture presented here may help form the basis for a deeper understanding of the properties of all kinds of adsorbed films.

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VOLTAGE-INDUCED VORTICITY AND OPTICAL FOCUSING IN LIQUID CRYSTALS

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We have observed that liquid crystals of the *p*-azoxyanisole type exhibit macroscopic rotational motion above a threshold voltage. The vortex motion of the birefringent liquid results in a lattice of cylindrical lenses whose focal lengths are voltage variable. These experiments are in excellent agreement with the dynamic predictions of the continuum theory.

The effects of electric fields on the optical properties of liquid crystals have received considerable attention recently because of their display applications.[1] We report experiments involving the dynamic scattering mode (DSM)[2] which is observed in certain nematic liquid crystals. Electric fields applied to *p*-azoxyanisole (PAA) induce a visible pattern, referred to as a domain pattern.[3] Our experiments are the first to demonstrate that the visible pattern results from the formation of liquid crystal lenses concomitant with cells of liquid rotation. The experiments also provide the first complete experimental description of the threshold voltage for vortex formation. We believe that these observations give conclusive verification of Helfrich's conduction-induced-alignment theory.[4]

We have used the customary experimental configuration in which the liquid crystal is held in a parallel plate capacitor with transparent Nesatron electrodes. The sample thickness d was held fixed by Teflon spacers. PAA was purchased commerically and used without further purification. The sample was viewed with a Leitz polarizing microscope in the transmission mode. A hot stage regulated the temperature in the liquid crystal region, 116° to 136°C for PAA. Both Nesatron surfaces were wiped vigorously along a single direction in order to promote a uniform alignment of the order director $\vec{S}$.[5] In the continuum theory of liquid crystals, $\vec{S}$ is a unit vector pointing in the average direction of molecular order in a fluid volume element.[6] An orthogonal coordinate system is established by defining the x direction as the rubbing direction and the z direction as the axis of the applied electric field and the gravitational field. The domain lines can be observed using either dc or audio-frequency ac electric fields. We have found that the electrical and optical properties are much more reproducible when ac fields are employed. dc fields may unnecessarily complicate the experiments because of impurity-ion polarization and electrochemical reactions at the electrodes. The